THE COMPLEX PROJECTIVE SPACE $\mathbb{CP}^1$

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Hyperbolic geometry appeared in part as an attempt to understand linear transformations.

We start by working in $\mathbb{R}^2$ and looking at all straight lines through the origin. Placing a horisontal "affine line" at $y = 1$, we see that every line through the origin intersects the affine line at a unique point. The only line through the origin for which this doesn't work is the line $y = 0$.

Associating the lines with points on the affine line, we get that the vertical line is associated with $x = 0$, the line $y = x$ is associated with $x = 1$, and so on. The smaller the slope of the line (as long as we keep it positive), the higher the x-coordinate associated with it. As we approach the horisontal line, the associated number approaches infinity. Thus it makes sense to associate the line $y = 0$ with a newly-defined point "∞". Now, if we keep reducing the slope of the line (so it's now negative), the intersection with the affine line occurs at negative x values and eventually returns to our original vertical line. Thus, by adding the "ideal point", we in effect connected the two "ends" of the affine line. If we take two points at $x = 0$ and move one in the positive x direction, and the other in the negative x direction, they will actually meet at the "point at infinity".

Thus, we can represent the lines through the origin in $\mathbb{R}^2$ by the points on the affine line, and define the point "∞" as the point corresponding to the the line y=0. This new set, $\mathbb{R} + \{\infty\}$ is called the real projective space, $\mathbb{RP}^1$.

Similarly, we can take $\mathbb{R}^3$ and put an affine *plane* at $z = 1$. Each line through the origin corresponds to a point on this plane, except for lines in the x-y plane. These remaining lines can be represented by a "line at infinity" (a real projective line, $\mathbb{RP}^1$). This space, consisting of $\mathbb{R}^2 + (\mathbb{RP}^1$ ideal points) is denoted $\mathbb{RP}^2$.

For now, though, we'll work in a somewhat simpler space denoted $\mathbb{CP}^1$. It's defined similarly to $\mathbb{RP}^1$. We again start with a two-dimensional

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space, in this case $\mathbb{C}^2$, and look at all the lines through the origin. As a result, we need $\mathbb{C} + \{\infty\}$ to represent this space. While discussing $\mathbb{RP}^1$ we mentioned that the two ends of it are connected by the point at infinity, creating something like an infinitely big circle. The edges of $\mathbb{CP}^1$ are also connected to each other by the point at infinity: no matter in which direction you go, you end up at infinity. Because of this, $\mathbb{CP}^1$ is also known as a "Riemann sphere".

Although we can make models of hyperbolic space in various ways, it turns out that $\mathbb{CP}^1$ is the simplest space to work with.

So now we have some justification for studying $\mathbb{CP}^1$. Although it's useful to remember that we're constantly dealing with objects in $\mathbb{C}^2$, for now we will ignore that fact and focus on the properties of the Möbius sphere. We can represent it on paper or in a computer program as just $\mathbb{C}$, keeping in mind that the point at infinity is also part of the space. So, for the most part, things stay the same when we add the extra point. However, if we follow what would be a straight line in $\mathbb{C}$, it turns out that we go off to infinity, "loop around it" and come back from the other side of the line. Furthermore, two straight lines can't be "parallel" (which is defined as non-intersecting) since any two straight lines go through infinity and must therefore intersect there. In fact, most straight lines intersect in two places - their intersection in $\mathbb{C}$ and at infinity. So lines behave very much like circles, so much so that they can actually be thought of as "circles that go through ∞".

Based on this, we can define a new object in $\mathbb{CP}^1$:
A **chain** in $\mathbb{CP}^1$ is either a Euclidean straight line or a Euclidean circle.

The next handout will discuss chains and "homographies", which are transformations that preserve chains.

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