CHAINS IN $\mathbb{CP}^1$

BILL GOLDMAN

A *chain* in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is either a Euclidean straight line or a Euclidean circle. The real axis, or more accurately, $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ is a chain. Furthermore $\mathsf{PGL}(2,\mathbb{C})$ acts transitively on chains with isotropy group $\mathsf{PGL}(2,\mathbb{R})$.

Chains can be efficiently manipulated in terms of the matrices representing their inversions. Complex conjugation

$$\iota_{\mathbb{R}} : z \mapsto \bar{z}$$

is inversion in $\mathbb{RP}^1$, and is an *anti-involution* of $\mathbb{CP}^1$. More generally, if $c \subset \mathbb{CP}^1$ is a chain, and ι_c is inversion in c, then the composition $\iota_c \circ \iota_{\mathbb{R}}$ is a projective automorphism of $\mathbb{CP}^1$. Thus an anti-involution will be represented by the composition of a projective automorphism of $\mathbb{CP}^1$ with complex conjugation:

$$\phi : z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} \tag{1}$$

where $a, b, c, d \in \mathbb{C}$. We may assume as usual that $ad - bc = 1$.

Denote the set of fixed points of a transformation ϕ by $\mathsf{Fix}(\phi)$. We shall write

$$c = \mathsf{Fix}(\phi_c)$$

where c is a chain.

Writing

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{SL}(2,\mathbb{C}) \tag{2}$$

The corresponding anti-projective transformation $\phi_A = \phi$ defined by (**??**) has order two if and only if

$$A\bar{A} = \pm 1.$$

Exercise 1. *Prove that* $\mathsf{Fix}(\phi_A) \neq \emptyset$ *if and only if*

$$A\bar{A} = 1 \tag{3}$$

and that ϕ_A *is fixed-point-free if and only if* $A\bar{A} = -1$. *If* A *is given by* (**??**), *then* (**??**) *correspond to the conditions that* a, d *are complex-conjugate of one another and* c, d *are purely imaginary. Furthermore*

Date: September 24, 2004.

the matrix $A \in \mathsf{SL}(2,\mathbb{C})$ *is determined uniquely up to multiplication by* ± 1*.*

An example of a free anti-involution is

$$\phi_A : z \mapsto -\frac{1}{\bar{z}}$$

(which corresponds to the antipodal map under stereographic projection). The corresponding matrix is

$$A = \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Inversion in the unit circle is the anti-involution

$$\phi_A : z \mapsto \frac{1}{\bar{z}}$$

corresponding to the matrix

$$A = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Suppose c is a *finite chain,* that is, a Euclidean circle. Then ϕ_c maps ∞ to its (Euclidean) center z_0. Thus the corresponding linear transformation A maps the line $\mathbb{C} \times \{0\}$ (corresponding to $\infty \in \mathbb{CP}^1$) to the line containing $\begin{bmatrix} z_0 \\ 1 \end{bmatrix}$.

Now let $B \in \mathsf{GL}(2,\mathbb{C})$ be a matrix representing a linear transformation of $\mathbb{C}^2$. Since the anti-involution fixing $B(c)$ is the composition

$$z \xmapsto{B^{-1}} B^{-1}(z) \xmapsto{A} A\overline{B^{-1}(z)} \xmapsto{B} BA\overline{B^{-1}(z)} = BA\bar{B}^{-1}(\bar{z}),$$

$\mathsf{PSL}(2,\mathbb{C})$ acts on anti-involutions by:

$$B : A \longmapsto BA\bar{B}^{-1}. \tag{5}$$

From this we derive the formula for the anti-involutions corresponding to Euclidean circles. Translation by $z_0 \in \mathbb{C}$

$$z \longmapsto z + z_0$$

corresponds to the matrix

$$B_{z_0} = \begin{bmatrix} 1 & z_0 \\ 0 & 1 \end{bmatrix}$$

and dilation by R^2

$$D_{R^2} : z \longmapsto R^2 z$$

corresponds to the matrix

$$B = \begin{bmatrix} R & 0 \\ 0 & R^{-1} \end{bmatrix}.$$

Taking A to correspond to the unit-circle (**??**), we see that the matrix

$$A_{z_0,R} := iR^{-1} \begin{bmatrix} z_0 & R^2 - z_0\overline{z_0} \\ 1 & -\overline{z_0} \end{bmatrix} \tag{6}$$

corresponds to the Euclidean circle centered at z_0 having radius R.

An *infinite chain* is a chain containing ∞ and is represented by a Euclidean straight line.

Exercise 2. *Let $A \in \mathsf{SL}(2,\mathbb{C})$ be given by* (**??**)*.*

$$\phi_A(\infty) = \frac{a}{c}.$$

Deduce that infinite chains c correspond to upper-triangular matrices satisfying (**??**)*.*

The *real axis* $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ corresponds to the anti-involution

$$z \longmapsto \bar{z}$$

corresponding to the identity matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and rotation by θ

$$z \longmapsto e^{i\theta} z$$

corresponds to the matrix

$$B = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$$

so that the straight line $e^{i\theta}\mathbb{R}$ of inclination θ corresponds to the anti-involution

$$z \longmapsto e^{2i\theta}\bar{z}$$

with corresponding matrix

$$A_\theta := \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Thus the chains passing through $0, \infty$ correspond to

$$\{A_\theta \mid \theta \in [0.\pi].$$

The general infinite chain is obtained by translation. Since a Euclidean straght line not containing 0 is completely determined by its *closest-point* (to 0), which lies in $\mathbb{C}^* := \mathbb{C} - \{0\}$, the infinite chains not containing 0 are parametrized by $\mathbb{C}^*$.

Exercise 3. *Show that the infinite chain corresponding to* $z_1 \in \mathbb{C}^*$ *corresponds to the upper-triangular matrix*

$$A = |z_1|^{-1} \begin{bmatrix} iz_1 & -2iz_1\bar{z}_1 \\ 0 & -i\bar{z}_1 \end{bmatrix}.$$

Describe a topological model for the set of all Euclidean straight lines (infinite chains) as the blow-up *of* $\mathbb{R}^2$ *at the origin.*

Thus *homogeneous space of chains* is stratified into three strata diffeomorphic to S^1, $\mathbb{C}^*$ and $\mathbb{C} \times \mathbb{R}^+$ respectively.

Exercise 4. *This homogeneous space can be described in many different ways, such as* $\mathsf{GL}(2,\mathbb{C})/\mathsf{GL}(2,\mathbb{R})$, $\mathsf{PGL}(2,\mathbb{C})/\mathsf{PGL}(2,\mathbb{R})$, *etc. (Are these the same as* $\mathsf{PSL}(2,\mathbb{C})/\mathsf{PSL}(2,\mathbb{R})$*?)*

Since $\mathsf{GL}(2,\mathbb{R})$ *and* $\mathsf{U}(1,1)$ *are conjugate subgroups of* $\mathsf{GL}(2,\mathbb{C})$, *this homogeneous space admits alternate descriptions as* $\mathsf{GL}(2,\mathbb{C})/\mathsf{U}(1,1)$, $\mathsf{PGL}(2,\mathbb{C})/\mathsf{PU}(1,1)$, $\mathsf{SL}(2,\mathbb{C})/\mathsf{SU}(1,1)$, *etc.*

In terms of real (indefinite) orthogonal groups, this homogeneous space is $\mathsf{O}(3,1)/\mathsf{O}(2,1)$.

Exercise 5. *Let* $A_1, A_2 \in \mathsf{GL}(2,\mathbb{C})$ *satisfy* (**??**) *with corresponding chains* $c_i := \mathsf{Fix}(\phi_{A_i})$. *Relate the angle of intersection* $\angle(c_1c_2)$ *to the trace* $\mathsf{tr}(A_1A_2)$. *Define an* imaginary angle of intersection *when* $c_1 \cap c_2 = \emptyset$. *What is the condition that two chains be orthogonal?*

Exercise 6. *Three pairwise distinct and tangent chains possess a unique common orthogonal chain.Given three distinct chains, determine the condition (in terms of matrices satisfying* (**??**)*) that three chains possess a common orthogonal chain.*

Exercise 7. *Orbits of matrices satisfying* (**??**) *under scalar multiplication by* $\pm$ *correspond to chains. Show that the matrices themeselves correspond to* oriented chains. *An orientation an a chain c corresponds to one of the two discs comprising the complement of c.*

An unordered pair of distinct points is the fixed-point-set of an involution. For example the involution fixing $0, \infty$ is $z \longmapsto -z$ and the involution fixing ± 1 is $z \longmapsto 1/z$.

Exercise 8. *Involutions correspond to matrices (defined up to* ± 1*) satisfying* $A^2 = -1$. *Show that the matrix corresponding to the pair*

$\{\infty, z_1\}$ is

$$\begin{bmatrix} -1 & 2z_1 \\ 0 & 1 \end{bmatrix}$$

and that the matrix corresponding a pair $\{z_1, z_2\} \subset \mathbb{C}$ with $z_1 \neq z_2$ is

$$\frac{1}{z_1 - z_2} \begin{bmatrix} 1 & -z_1 \\ 1 & -z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -z_2 & z_1 \\ -1 & 1 \end{bmatrix} = \frac{1}{z_1 - z_2} \begin{bmatrix} -(z_1 + z_2) & 2z_1 \\ -2z_2 & z_1 + z_2 \end{bmatrix}$$

For example the matrix corresponding to $\{\pm 1\}$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Define data types (analogous to those for chains) for unordered pairs of points, and also ordered pairs of points. Describe the action of $\mathsf{PGL}(2, \mathbb{C})$ *(as well as the anti-projective automorphisms of $\mathbb{CP}^1$) on pairs.*

What is the condition that a chain and a pair be incident? Define a data type for a circular arc (including line segments and complements of line segments in infinite chains).

Experimental Geometry Lab, 3113 Mathematics Building, Math Dept., University of Maryland, College Park, MD 20742

E-mail address: `wmg@math.umd.edu`