Euclidean, Spherical and Hyperbolic Shadows

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Consider this familiar related rates problem from freshman calculus:

> ([7] pg 181) A street light is mounted at the top of a 15-ft tall pole. A man 6 feet tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

The standard solution uses similar triangles, a feature uniquely Euclidean, but it can be solved another way using intersections of lines. The same question can then be posed and solved in any geometry where we can compute intersections of lines. Here we solve the problem for an individual walking in the elliptic or hyperbolic plane. Working in the other geometries illustrates key differences between the flat geometry of Euclidean space and these other constant curvature, two dimensional geometries.

The setup

Let the individual have fixed height h and the lamppost have fixed height l (of course assuming $h < l$). Let s be the length of the shadow and d the distance from the base of the lamppost to the individual's feet. A good calculus student will draw the picture in Figure 1.

Solving the problem requires that we write s as a function of d. In the Euclidean plane, the ground is a straight line, as is the light ray from the lamp

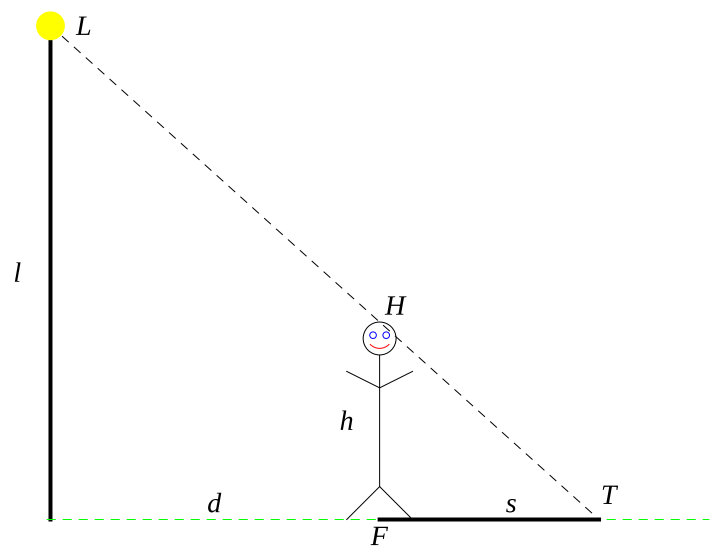

Figure 1: The setup in the Euclidean plane.

through the top of the individual's head. In the other two plane geometries we draw similar schematic pictures. We assume that the ground is flat and the light ray travels along a straight path, that is both are distance minimizing curves in the appropriate geometry. Such a curve is called a *geodesic*.

Figure 2 is the setup in spherical geometry, shown both on the sphere and after stereographic projection to the plane. Figure 3 gives the setup in two models of hyperbolic geometry, the Poincaré unit disk and the upper half plane ([1] describes these models).

We compute s as a function of d in each geometry. The result is a fantastic illustration of the effect curvature has on these, and nicely illustrates the duality between spherical and hyperbolic geometry.

Before reading the remainder of this article, the reader might enjoy examining figures 2 and 3 to get some intuition for what to expect from the shadow in each geometry. If imagination is not enough, the animations available online at [3] should convince you that the shadow on the sphere is periodic, while in the hyperbolic plane the shadow length grows to infinity very

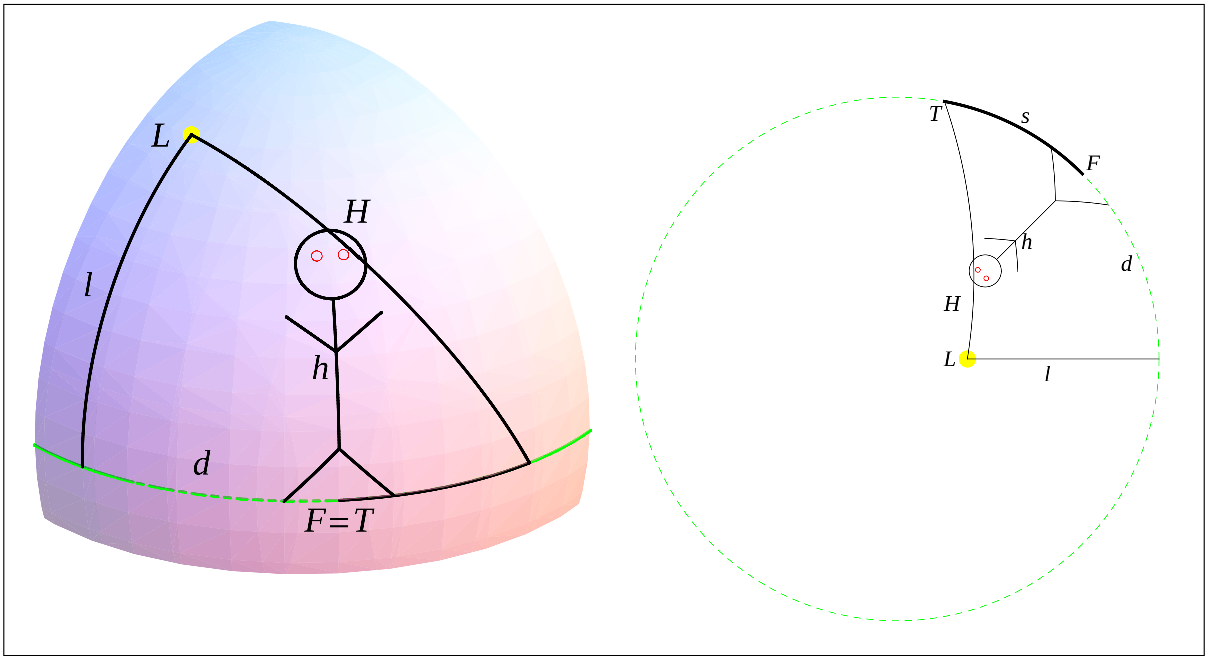

Figure 2: The setup in spherical geometry.

rapidly. The computations are a bit complicated, but the pictures make the complicated formulas believable.

Euclidean Shadows

To solve the problem, we are led to write down the shadow length s as a function of Stickman's distance d from the base of the light post. We assume that the ground lies along the x-axis and the light post along the y-axis. This enables us to write the equation of the light ray from the light source through the top of Stickman's head.

$$y = \left(\frac{h-l}{d}\right) x + l$$

The tip of the shadow is the x-intercept of this line, $\left(\frac{ld}{l-h}, 0\right)$. The shadow length is simply the distance from the shadow tip to Stickman's feet, i.e.

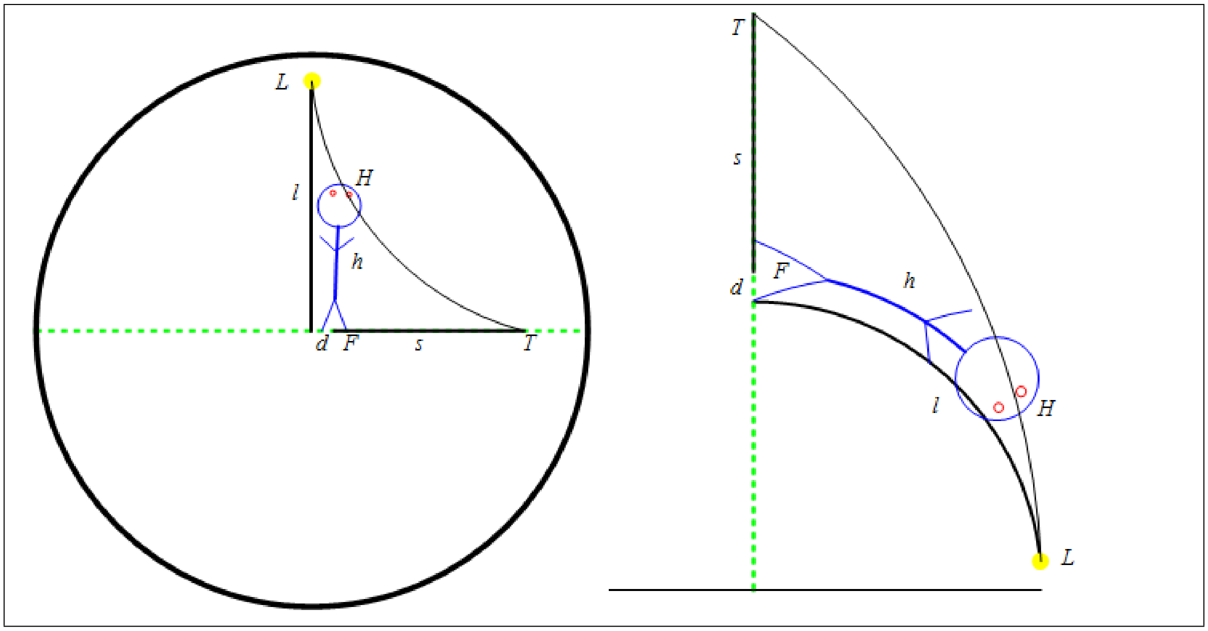

Figure 3: The setup in hyperbolic geometry.

$$\begin{aligned} s &= \text{dist(feet,shadowtip)} \\ &= \text{dist}\left((d,0)\,,\left(\frac{ld}{l-h},0\right)\right). \\ &= d\left(\frac{h}{l-h}\right) \end{aligned}$$

We see that shadow length is directly proportional to the distance walked. The shadow becomes arbitrarily long, but always has finite length. The problem can be finished by differentiating to obtain the rate of change of shadow length and adding it to the speed of the individual moving on the ground.

Spherical Shadows

Let's now compute s as a function of d assuming Stickman walks in spherical geometry. The standard model for spherical geometry is the unit sphere centered at the origin in $\mathbb{R}^3$. Here geodesics are *great circles*, that is the intersection of the unit sphere with linear subspaces of $\mathbb{R}^3$. Points in spherical geometry are represented by unit vectors in $\mathbb{R}^3$ and the Euclidean metric on $\mathbb{R}^3$ induces a metric on the sphere. The spherical distance between two points is simply the Euclidean arc length along the great circular arc connecting these points. Let $\mathbb{SD}(u, v)$ denote the spherical distance between unit vectors u and v, which is related to the dot product by

$$\cos\left(\mathbb{SD}(u, v)\right) = u \cdot v.$$

We assume that the ground is the sphere's equator and the light post lies along a meridian. Observe that the ground then has finite length 2π and the farthest the light can be away from the ground is $\frac{\pi}{2}$. Even before working out the computation, Figure 4 makes it clear that there is a key difference between $l = \frac{\pi}{2}$ and $l < \frac{\pi}{2}$. Specifically, when $l = \frac{\pi}{2}$, the light ray intersects the top of Stickman's head at a right angle, then runs along his body, so no shadow is produced. He is always standing directly under the light, no matter where he walks. Figure 4 also suggests that if $l < \frac{\pi}{2}$, the shadow length will be periodic, not surprising as the whole geometry is finite.

Now for some coordinates. Suppose the base of the lamp is at $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If the light post lies along the meridian through that point and has height l, then the light source is at $L = \begin{pmatrix} \cos(l) \\ 0 \\ \sin(l) \end{pmatrix}$. After walking distance d from the base of the light post then Stickman's feet are at $F = \begin{pmatrix} \cos(d) \\ \sin(d) \\ 0 \end{pmatrix}$. Assuming

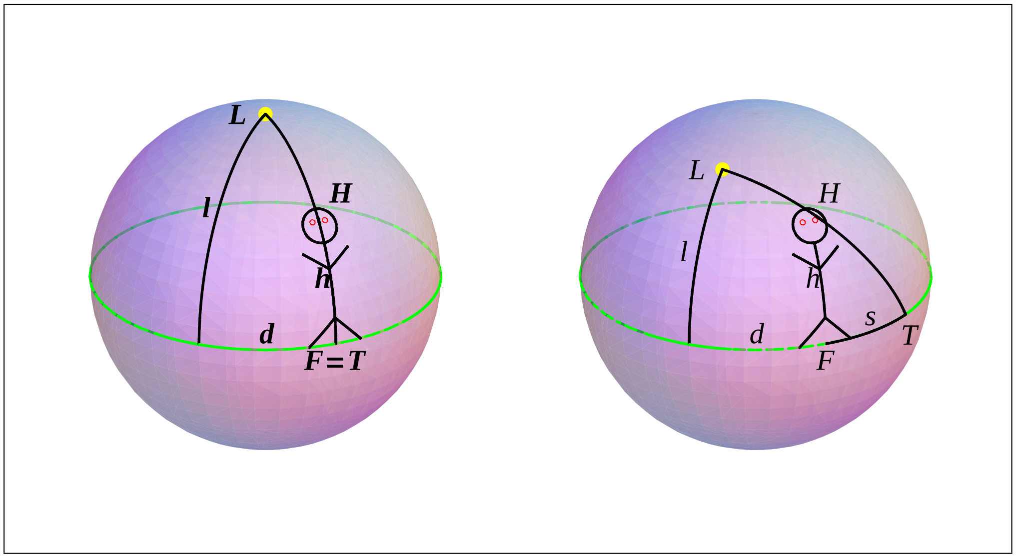

Figure 4: Stickman on the sphere.

Stickman has height h, the top of his head is at $H = \begin{pmatrix} \cos(d)\cos(h) \\ \sin(d)\cos(h) \\ \sin(h) \end{pmatrix}$.

We can now determine the geodesic light ray which forms the shadow. The light source L and the top of Stickman's head H are unit vectors in $\mathbb{R}^3$ which determine a unique plane through the origin. The intersection of that plane with the unit sphere is the geodesic light ray. The shadow tip is the intersection of this plane with the equator. The tip of the shadow, T, is determined by the fact that it lies along the intersection of the plane containing L and H and the equatorial plane - and the fact that it is a unit vector. Its coordinates are found by solving simultaneously $(L \times H) \cdot T = 0$, $\|T\| = 1$ and $z = 0$,

This system has 2 antipodal solutions. With the aid of a computer algebra system we obtain:

$$T = \begin{pmatrix} \frac{\cos(d)\cos(h)\sin(l)-\sin(h)\cos(l)}{\sqrt{-2\cos(d)\sin(h)\cos(h)\sin(l)\cos(l)+\sin^2(h)\cos^2(l)+\cos^2(h)\sin^2(l)}} \\ \frac{\sin(d)\cos(h)\sin(l)}{\sqrt{-2\cos(d)\sin(h)\cos(h)\sin(l)\cos(l)+\sin^2(h)\cos^2(l)+\cos^2(h)\sin^2(l)}} \\ 0 \end{pmatrix}.$$

The shadow length s is the spherical distance from the Stickman's feet F to the tip of the shadow T. We find that

$$\begin{aligned} \cos(s) &= F \cdot T \\ &= \frac{\cos(h)\sin(l) - \cos(d)\sin(h)\cos(l)}{\sqrt{-2\cos(d)\sin(h)\cos(h)\sin(l)\cos(l) + \sin^2(h)\cos^2(l) + \cos^2(h)\sin^2(l)}} \end{aligned}$$

This is indeed complicated and scary, but note that if $l = \frac{\pi}{2}$ the entire expression simplifies to $\cos(s) = 1$. In this case shadow length is identically 0, as expected. Figure 5 graphs s as a function of d for fixed light height $l = \frac{\pi}{3}$ and various choices for h. The shadow length is periodic. What is less obvious is that as $h \to l$ the relationship between s and d approaches a linear relationship, though not uniformly.

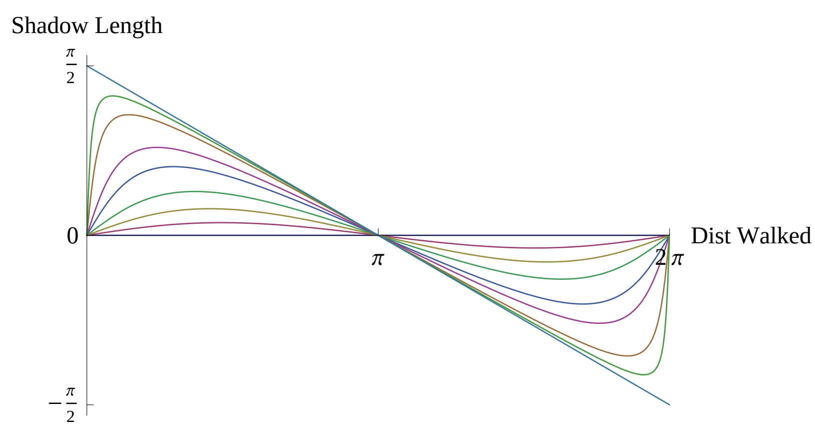

Figure 5: Spherical shadows lengths.

Hyperbolic Shadows

What if Stickman is walking in the hyperbolic plane? For readers unfamiliar with this geometry there are a myriad of great references. Specifically [1], [4] and [5] are excellent introductions. The many different models of the hyperbolic plane often present a challenge when first learning the subject as Euclidean and spherical geometry each have a single standard model. The relationships between different models however is key to understanding hyperbolic geometry as each model has its own advantages & disadvantages. Anderson ([1]) does a particularly good job relating the various models.

For computational ease, we work in the upper half plane model. This is the upper half of the complex plane $\{x + iy : y > 0\}$. The geodesics are either vertical rays or circular arcs intersecting the real axis at a right angle.

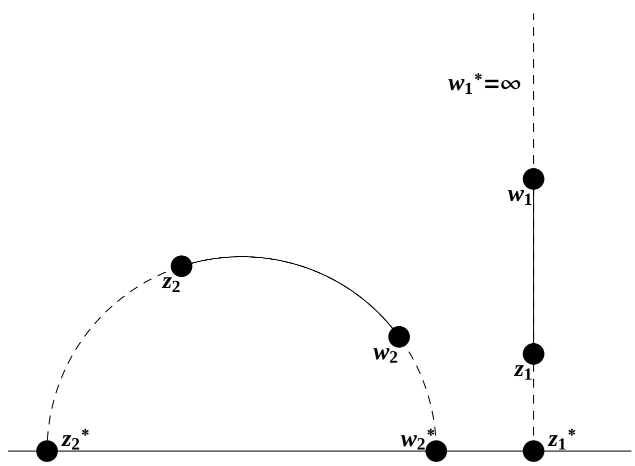

Figure 6: Typical geodesics in the upper half plane.

For the shadow problem, we need a formula for distances. Given distinct points z and w in the upper half plane we connect these with a geodesic segment. If $\text{Re}(z) = \text{Re}(w)$ this is the vertical line segment connecting z and w, otherwise it is a circular arc. Extend this geodesic segment to a complete geodesic and observe that this intersects $\mathbb{R} \cup \{\infty\}$ twice. The endpoints are called *ideal points* and we denote them by z^* and w^*, so that z is between z^* and w. If the geodesic is a vertical line, one of the ideal endpoints is ∞ (see figure 6). Let $\mathbb{HD}(z,w)$ denote the hyperbolic distance between z and w, which by definition is

$$\mathbb{HD}(z,w) = \begin{cases} \ln\left(\frac{(z-w^*)(w-z^*)}{(w^*-w)(z^*-z)}\right) & \text{if neither } z^* \text{ nor } w^* \text{ is } \infty \\ \ln\left(\frac{\mathrm{Im}(z)}{\mathrm{Im}(w)}\right) & \text{if } z^* = \infty \\ \ln\left(\frac{\mathrm{Im}(w)}{\mathrm{Im}(z)}\right) & \text{if } w^* = \infty \end{cases} \tag{1}$$

.

To calculate s as a function of d, assume the ground is the positive imaginary axis and the base of the light post is at i. If the light post is perpendicular to the ground then it lies along the the unit circle. The lamp then has a distance l from i, and this is the point $L = \tanh(l) + i\ \mathrm{sech}(l)$. After walking along the ground a distance d, Stickman's feet are at $F = ie^d$. Assuming he is standing upright means his body lies along the circular arc centered at the origin of radius e^d. Since his height is h, the top of his head is $H = e^d(\tanh(h) + i\ \mathrm{sech}(h))$. The reader is encouraged to use 1 to check that $\mathbb{HD}(i,F) = d$, $\mathbb{HD}(i,L) = l$ and $\mathbb{HD}(F,H) = h$. In fact an excellent exercise would be to derive these coordinates yourself (Hint: The lamp post has its base at i and the ideal points of that geodesic are -1 and 1).

Let T denote the tip of the shadow, the intersection of the geodesic light ray with the ground. The light ray is the geodesic arc through L and H, which we can explicitly compute. The fact that it intersects the real axis at a right angle means that the circle is centered at some point on the real axis, say x_0. If r is its radius, then this geodesic satisfies the equation

$$(x - x_0)^2 + y^2 = r^2.$$

Substituting in the real and imaginary parts of L and H, we obtain a system with unknowns x_0 and r

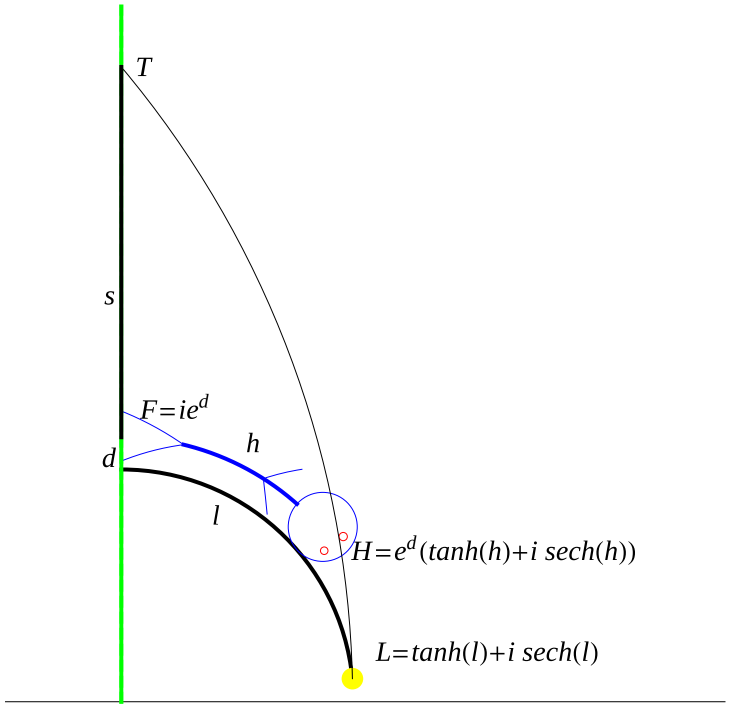

Figure 7: The setup in the upper half plane.

$$(\tanh(l) - x_0)^2 + (\ \mathrm{sech}(l))^2 = r^2$$
$$(e^d \tanh(h) - x_0)^2 + (e^d\ \mathrm{sech}(h))^2 = r^2,$$

which we can solve for x_0 and r. With the aid of a computer algebra system we obtain

$$T = i\sqrt{\frac{e^d\left(\tanh(h) - e^d \tanh(l)\right)}{e^d \tanh(h) - \tanh(l)}}.$$

To compute s we simply need the distance from F to T. Since these points both lie on the imaginary axis the distance is

$$s = \ln\left(e^{-d}\sqrt{\frac{e^d\left(\tanh(h) - e^d \tanh(l)\right)}{e^d \tanh(h) - \tanh(l)}}\right) \tag{2}$$

.

We now have the desired relationship between s and d. This looks complicated but a brief analysis confirms what we might already suspect. Note that s is undefined if the numerator or denominator of the radicand becomes zero. This occurs when

$$d = d_0 = \pm \ln \left(\frac{\tanh(l)}{\tanh h} \right).$$

This has a wonderful interpretation. After walking a finite distance d_0 in either direction Stickman's shadow has infinite length! Figure 8 illustrates this. When $d = d_0$ the hyperbolic ray forming the shadow becomes a vertical line which does not intersect the imaginary axis (ground). When $d = -d_0$ the ray will becomes a circular arc whose endpoint is the origin which also does not intersect the positive imaginary axis. What a sharp contrast with the Euclidean case, where the shadow always is finite.

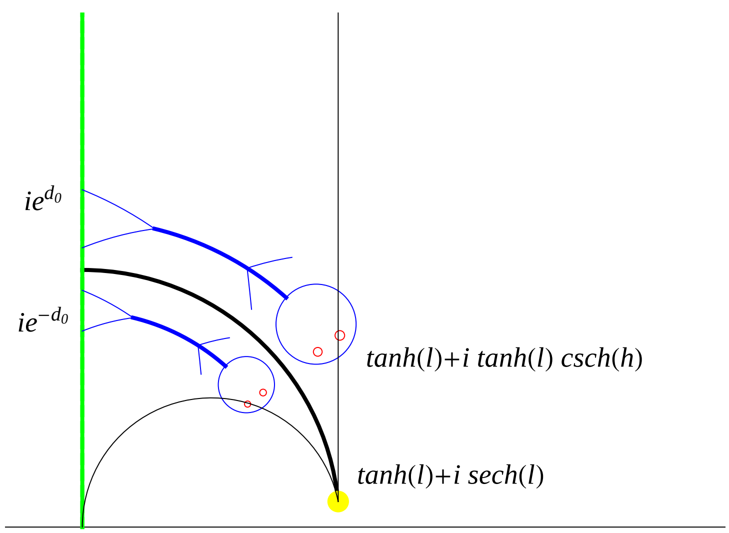

Figure 8: A finite distance from the lamp, Stickman's shadow becomes infinite.

Duality

We conclude with a comparison of these seemingly unrelated geometries. Figure 9 graphs the solutions for each geometry for fixed l and h satisfying $0 < h < l < \frac{\pi}{2}$.

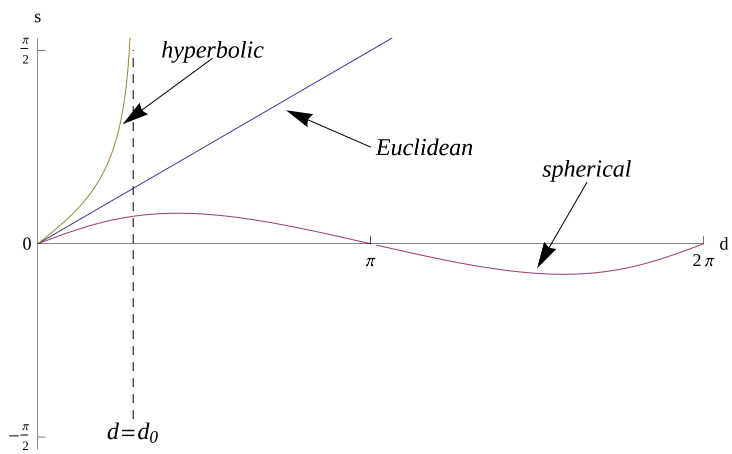

Figure 9: Shadow length s as a function of d in all three geometries.

Recall that the solution in the hyperbolic case was

$$s = \ln\left(e^{-d}\sqrt{\frac{e^d\left(\tanh(h) - e^d\tanh(l)\right)}{e^d\tanh(h) - \tanh(l)}}\right).$$

Taking cosh of both sides and then applying a handful of hyperbolic trigonometric identities, we rewrite this as

$$\cosh(s) = \frac{\cosh(d)\sinh(h)\cosh(l) - \cosh(h)\sinh(l)}{\sqrt{\cosh^2(l)\sinh^2(h) - 2\cosh(d)\cosh(h)\cosh(l)\sinh(h)\sinh(l) + \cosh^2(h)\sinh^2(l)}}.$$

Compare this with the solution obtained in the spherical case which was

$$\cos(s) = \frac{\cos(h)\sin(l) - \cos(d)\sin(h)\cos(l)}{\sqrt{\cos^2(l)\sin^2(h) - 2\cos(d)\sin(h)\cos(h)\sin(l)\cos(l) + \cos^2(h)\sin^2(l)}}.$$

These formulas make dramatically clear the duality between spherical and hyperbolic geometry. In particular since $\cosh(i\theta) = \cos(\theta)$ and $\sinh(i\theta) = i\sin(\theta)$, we can transform the hyperbolic relationship into the spherical one by multiplying all variables by $-i$. It can be shown that if all variables are small, the solutions are approximately Euclidean. In other words, the solutions agree up to first order. Stickman should be assured that if his journey is in a direction with curvature close to zero and he doesn't walk too far, his shadow will behave approximately as in flat Euclidean space.

Final Remark

My own motivation for this came after answering the original question for a calculus class, then realizing that our solution using similar triangles was only applicable in Euclidean geometry. When teaching an undergraduate course one should pay attention when unnecessarily making implicit use of Euclidean geometry. Many classical problems from pre-calculus and calculus use facts about triangles or Euclidean trigonometry, and you can answer the same questions in spherical or hyperbolic trigonometry. I don't advise confusing students first learning calculus with these issues, but exciting investigations are there to be found by changing the geometry.

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