The space of chains in $\mathbb{C}P^1$ is a 3-manifold homeomorphic to $(S^2 \times \mathbb{R})/\mathbb{Z}_2$

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July 30, 2009

Definitions

Definition

$\mathbb{C}P^1$ is the Complex projective plane. We can think of it as $\mathbb{C} \cup \{\infty\}$ or the Riemann sphere.

Definition

A *chain* in $\mathbb{C}P^1$ is a circle on the Riemann sphere. The set of chains in $\mathbb{C}P^1$ corresponds to the set of circles and lines in $\mathbb{C}$ where a line in $\mathbb{C}$ can be considered a circle that goes through the point ∞ (the North pole of the sphere).

Definitions

Definition

A *signed chain* is an element of $\mathrm{Chains} \times \mathbb{Z}_2$.

Definition

For $z \in \mathbb{C}P^1$, we define the *homogeneous coordinates* of z to be $\begin{pmatrix} z \\ 1 \end{pmatrix}$ if $z \in \mathbb{C}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if $z = \infty$.[1]

[1] Technically, homegenous coordinates are defined only up to a non-zero complex scalar, $v \sim z_0 v$, $z_0 \in \mathbb{C}_*$

Notation

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$\mathcal{H}$ is the set of 2×2 Hermitian matrices over $\mathbb{C}$.

Notation

$\mathcal{H}_{<0}$ is the Hermitians with negative determinant.

Notation

$\hat{\mathcal{H}}$ is $\mathcal{H} \setminus I_2$ where I_2 is the 2×2 identity matrix.

Representing Chains by Hermitian Matrices

Each $H \in \mathcal{H}$ defines a Hermitian form f_H by

$$f_H = \begin{pmatrix} z \\ 1 \end{pmatrix}^t \begin{pmatrix} a & w \\ \overline{w} & b \end{pmatrix} \begin{pmatrix} \overline{z} \\ 1 \end{pmatrix}$$

$$= az\overline{z} + wz + \overline{wz} + b$$

We obtain (possibly empty) point sets in $\mathbb{C}P^1$ by setting $f_H = 0$.

Representing Chains by Hermitian Matrices

Claim

The set of chains in $\mathbb{C}P^1$ is the set $\mathcal{H}_{<0}/\mathbb{R}_*$

Proof.

Note that for $a \in \mathbb{R}_*$,

$$f_{aH} = 0 \iff f_H = 0$$

so $H \sim aH$ as *chains*.

Circles in $\mathbb{C}$

Let $b = -|z_0|^2 - r^2$, $w = -\overline{z_0}$, $a = 1$, then

$$f_H = 0 \iff |z - z_0| = r^2$$

This gives us

$$H = \begin{pmatrix} a & -\overline{z_0} \\ -z_0 & -|z_0|^2 - r^2 \end{pmatrix}$$

Determinant is $-2|z_0|^2 + r^2 < 0$.
Conversely, given an element in $\mathcal{H}_{<0}/\mathbb{R}_*$ with $a \neq 0$, it determines a circle.

Lines in $\mathbb{C}$

Any line in $\mathbb{C}$ can be written

$$e^{i\theta} z + e^{-i\theta}\overline{z} + d = 0 \quad d \in \mathbb{R}$$

so

$$H = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & d \end{pmatrix}$$

Determinant is -1.
Conversely, any matrix with $a = 0$ determines a line. ■

Fun Facts About Mobius Transformations

1. Mobius transformations is the group of conformal automorphisms of $\mathbb{C}P^1$.
2. Form: $z \mapsto \frac{az+b}{cz+d}$.
3. $M \in \mathrm{GL}(2,\mathbb{C})$ acts $z \in \mathbb{C}P^1$ by Mz for z in homogeneous coordinates.
4. Group of Mobius transformations is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$.

Action of Mobius Transformations on Chains

Let $w = Mz$, and letting $H \in \mathcal{H}_{<0}/\mathbb{R}_*$

$$\begin{aligned} z^t H\overline{z} &= (M^{-1}w)^t H\overline{(M^{-1}w)} \\ &= w^t({M^{-1}}^t H\overline{M^{-1}})\overline{w} \end{aligned}$$

which is the hermitian form in z determined by the chain ${M^{-1}}^t H\overline{M^{-1}}$.[2]

[2]That the result is a chain needs proving. Proof is ommitted here due to time constraints

Two Definitions

Definition

A *homeomorphism* $f : X \to Y$, X, Y topological spaces is a continuous open bijection. (Open means it takes open sets to open sets).

Definition

A map $f : X \to Y$ is *G-equivariant* for a group G if f commutes with the action of G on X and Y. That is, $f(g \cdot x) = g \cdot f(x)$.

A homeomorphism between $\hat{\mathcal{H}}$ and $S^2 \times \mathbb{R}$

We now define a map $\psi : \hat{\mathcal{H}}/\mathbb{R}_+ \to S^2 \times \mathbb{R}$ by

$$\psi \begin{pmatrix} a & z \\ \bar{z} & d \end{pmatrix} = \left(\frac{1}{\sqrt{\left(\frac{a-d}{2}\right)^2 + |z|^2}} \begin{pmatrix} \frac{a-d}{2} \\ \Re(z) \\ \Im(z) \end{pmatrix}, \frac{a+d}{\sqrt{\left(\frac{a-d}{2}\right)^2 + |z|^2}} \right)$$

Claim

ψ is $\mathbb{R}_*$ equivariant.

Proof.

Let $\mathbb{R}_*$ act on $\hat{\mathcal{H}}$, $\lambda \cdot H = \lambda H$.
Let $\mathbb{R}_*$ act on $S^2 \times R$ by $\lambda \cdot (v,t) = (\sigma(\lambda)v, \sigma(\lambda)t)$ where $\sigma(\lambda)$ is the sign of λ.
The following diagram commutes:

$$\begin{array}{ccc} \hat{\mathcal{H}} & \xrightarrow{\psi} & S^2 \times \mathbb{R} \\ \mathbb{R}_* \downarrow & & \downarrow \mathbb{R}_* \\ \hat{\mathcal{H}} & \xrightarrow[\psi]{} & S^2 \times \mathbb{R} \end{array} \tag{1}$$

■

Claim

ψ *is bijective.*

Proof.

Let $(v, t) \in S^2 \times \mathbb{R}$, and let $v = (w, x, y)$ To find the pre-image of (v, t), we must solve the system of equations

$$\frac{w}{\sqrt{w^2 + x^2 + y^2}} = \frac{\left(\frac{a-d}{2}\right)}{\sqrt{\left(\frac{a-d}{2}\right)^2 + x^2 + y^2}}$$

$$\frac{x}{\sqrt{w^2 + x^2 + y^2}} = \frac{x}{\sqrt{w^2 + x^2 + y^2}}$$

$$\frac{y}{\sqrt{w^2 + x^2 + y^2}} = \frac{y}{\sqrt{w^2 + x^2 + y^2}}$$

$$\frac{t}{\sqrt{w^2 + x^2 + y^2}} = \frac{a + d}{\sqrt{w^2 + x^2 + y^2}}$$

which gives us

$$\psi^{-1}(v,t) = \begin{pmatrix} \frac{t+2w}{2} & x+iy \\ x-iy & \frac{t-2w}{2} \end{pmatrix}$$

By algebra, $\psi(H) = (v,t)$, so $(\psi \circ \psi^{-1})(v,t) = (v,t)$
Similar reasoning will convince us that $\psi^{-1} \circ \psi = id$ so the map is invertible and hence bijective. ■

Claim

ψ *is an open mapping.*

Proof.

Above inverse is continuous. ■

The Set of Signed Chains

Claim

Signed chains are $\mathcal{H}_{<0}/\mathbb{R}_+$.

Proof.

We defined chains as $(\mathcal{H}_{<0}/\mathbb{R}_*) \times \mathbb{Z}_2$.
$(\mathcal{H}_{<0}/\mathbb{R}_*) \times \mathbb{Z}_2 \simeq \mathcal{H}_{<0}/\mathbb{R}_+$ by mapping the equivalence class containing a matrix $H \in (\mathcal{H}_{<0}/\mathbb{R}_*) \times \mathbb{Z}_2$ to the equivalence class containing $H \in \mathcal{H}_{<0}/\mathbb{R}_+$.

∎

Restriction of ψ to Signed Chains

Claim

$\psi(\mathcal{H}_{<0}/\mathbb{R}_+) = S^2 \times (-2, 2)$.

Proof.

Let $(v, t) \in \psi(\mathcal{H}_{<0}/\mathbb{R}_+)$, $v = (w, x, y)$ Then

$$\psi^{-1}(v, t) = \begin{pmatrix} \frac{t+2w}{2} & x + iy \\ x - iy & \frac{t-2w}{2} \end{pmatrix}$$

By assumption, $\det\left(\psi^{-1}(v, t)\right) < 0$, so

$$
\begin{aligned}
&\left(\frac{t+2w}{2}\right)\left(\frac{t-2w}{2}\right) - x^2 - y^2 < 0 \\
\iff\ & \frac{t^2}{4} - w^2 - x^2 - y^2 < 0 \\
\iff\ & \frac{t^2}{4} < w^2 + x^2 + y^2 \\
\iff\ & t^2 < 4 \qquad\qquad (\text{Since } w^2 + x^2 + y^2 = 1) \\
\iff\ & t \in (-2, 2)
\end{aligned}
$$

■

Corollary

$\mathcal{H}_{<0}/\mathbb{R}_+ \simeq S^2 \times \mathbb{R}$.

Proof.

An immediate corollary of the above is $\mathcal{H}_{<0}/\mathbb{R}_+ \simeq S^2 \times (-2, 2)$. But $(-2, 2) \simeq \mathbb{R}$. ■

A Demonstration

We now have a homeomorphism from signed chains to $S^2 \times \mathbb{R}$. Let's see a demonstration of the signed chains coordinates.

Corollary

$\mathcal{H}_{<0}/\mathbb{R}_* \simeq (S^2 \times \mathbb{R})/\mathbb{Z}_2$

Proof.

$(\mathcal{H}_{<0}/\mathbb{R}_+)/\mathbb{Z}_2 \simeq (S^2 \times \mathbb{R})/\mathbb{Z}_2$ since $\mathbb{Z}_2$ action is free, proper and equivariant. ■

Additional Facts

1. There is a surjection from $(S^2 \times (-2,2))/\mathbb{Z}^2$ to $\mathbb{R}P^2$. Consider π_1
2. The fibers of $v \in \mathbb{R}P^2$ are Homeomorphic to $\mathbb{R}$. $\pi_1^{-1}(v) = v \times (-2,2) \simeq \mathbb{R}$