

Hyperbolic Toolbox Version 1.0

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Introduction

This toolbox is a set of *Mathematica* functions designed to:

- Draw hyperbolic geodesics and circles in the Poincaré unit disc as well as the Upper Half Plane model.
- Compute and apply linear fractional transformations of the isometry group $PSL(2, \mathbb{R})$.
- Map from the Upper Half Plane to the Unit Disc and vice versa.
- Allow the user to compute distances, label points, and other useful tools for creating detailed and interactive projects in hyperbolic geometry.

How to Use this Package

Place this file in the same directory as your project notebook. You can then load the package easily by inserting the following code at the beginning of your notebook:

```
SetDirectory[NotebookDirectory[]];  
  
<< HyperbolicToolbox_v1.0.m
```

Then go to Cell → Cell Properties and click on "Initialization Cell". This will ensure the package will be loaded when you evaluate your notebook.

Functions and their Usage

For each function, the type of input required and output generated will be listed. For example, $(\mathbb{H} \rightarrow \mathbb{D}, \text{Graphics})$ would indicate the function takes objects in the Upper Half Plane to the Unit Disc and generates graphics objects, while $(\mathbb{H} \rightarrow \mathbb{D})$ would indicate the function takes objects in the Upper Half Plane to the Unit Disc in the form of complex numbers.

Graphics Functions

Unless otherwise noted, all graphics functions should be used within the Graphics command.

- **PGSegment[{z1,z2}]**

($\mathbb{H} \rightarrow \mathbb{D}$, Graphics) Given two points from the Upper Half Plane, PGSegment draws hyperbolic lines on the disc model. An optional third argument specifies the minimum draw size.

- **UHPSegment[{z1,z2}]**

($\mathbb{H} \rightarrow \mathbb{H}$, Graphics) Draws hyperbolic lines in the Upper Half Plane.

- **PCircle[Center,Radius]**

($\mathbb{H} \rightarrow \mathbb{D}$, Graphics) Given a center in the Upper Half Plane and a radius, PCircle draws a hyperbolic circle on the Unit Disc. PCircle[Center, Radius, True] will draw the hyperbolic center as well as the circle.

- **UHPCircle[Center,Radius]**

($\mathbb{H} \rightarrow \mathbb{H}$, Graphics) Given a center in the Upper Half Plane and a radius, UHPCircle draws a hyperbolic circle in the Upper Half Plane. UHPCircle[Center, Radius, True] will draw the hyperbolic center as well as the circle.

- **UDPoint[z]**

($\mathbb{H} \rightarrow \mathbb{D}$, Graphics) Takes a point z in the Upper Half Plane and returns the Graphics primitive for this point in the Unit Disc.

- **UDText[z,text]**

(\mathbb{H} , Text $\rightarrow \mathbb{D}$, Text) places text in the Unit Disc according to the point z in the Upper Half Plane.

- **ShowP[List]**

Generates the Unit Disc and displays a list of Graphics primitives. Use in place of Graphics when drawing in the Poincaré disc.

- **Circumcircle**

Circumcircle draws the circle passing through three non-collinear points. The input can be six real numbers, three ordered pairs, or three complex numbers.

Mapping between the Upper Half Plane and the Unit Disc

- **ToUnitDisc[z]**
($\mathbb{H} \rightarrow \mathbb{D}$) maps the point z in the Upper Half Plane to the corresponding point in the Unit Disc.
- **ToUHP**
($\mathbb{D} \rightarrow \mathbb{H}$) or takes complex numbers or ordered pairs from the Unit Disk to the Upper Half Plane.
- **ToR2[z]**
($\mathbb{C} \rightarrow \mathbb{R}^2$) Returns an ordered pair from the real and imaginary parts of z . Use this function when a function requires an ordered pair instead of a complex number (such as most *Mathematica* Graphics primitives).

Linear Fractional Transformations

- **ApplyLFT[M,z]**
($\mathbb{H} \rightarrow \mathbb{H}$) Applies the linear fractional transformation corresponding to the 2×2 matrix M to the extended complex number z . If z is a list (or list of lists) then ApplyLFT applies the linear fraction across the list.
- **FindLFT[{z0, z1, zInf}, {w0, w1, wInf}]**
Returns a 2×2 matrix in $\text{PSL}(2, \mathbb{R})$ whose linear fractional transformation takes $z1$ to $w1$, $z2$ to $w2$, and $z3$ to $w3$. If the transformation is not in $\text{PSL}(2, \mathbb{R})$, it will return "ERROR."
- **InvolutionFixing[z1, z2]**
Returns the 2×2 matrix representing an involution fixing complex numbers $z1$, $z2$. It is normalized to have determinant - 1.
- **ReflectIn[z1,z2]**
Reflects the complex number z (upper half - plane coordinates) in the Poincaré geodesic with endpoints $z1$, $z2$ (extended real numbers). This is the effect of applying an anti - involution (anti - homography of order two) which fixes the chain orthogonal to the unit circle (real axis in upper half - plane coordinates) whose endpoints on the unit circle correspond to extended real numbers $z1$, $z2$.
- **SymmetryIn[z]**
Returns the involutory isometry of the Upper Half Plane fixing z .

Linear Fractional Transformations (continued)

- **FixedPointsSet[M]**
Returns extended real numbers, representing the attracting fixed point, then the repelling fixed point of the linear fractional transformation represented by the 2x2 matrix M.
- **Commutator[A,B]**
Returns the commutator of the matrices A and B.
- **CrossRatioMatrix[{z0,z1,zInf}]**
Returns the matrix whose linear fractional transformation takes z0 to 0, z1 to 1, zInf to ComplexInfinity.
- **HypElement[{x1,x2}]**
Returns some Hyperbolic Element in $PSL(2, \mathbb{R})$ with ideal fixed points x1 and x2.

Computing with Geodesics

- **HyperbolicLength[z1,z2]**
($\mathbb{H} \rightarrow \mathbb{R}$) Returns the length of the geodesic segment connecting z1 and z2.
- **MidpointEndpointsPGeodesic[z1, z2]**
($\mathbb{H} \rightarrow \mathbb{H}$) Returns the midpoint of the pair of endpoints of the Poincaré geodesic containing points z1 and z2 in the Upper Half Plane."
- **EndpointsPGeodesic[z1,z2]**
($\mathbb{H} \rightarrow \mathbb{H}$) Returns the endpoints of the Poincaré geodesic containing z1 and z2 in the Upper Half Plane.
- **IntersectionPoint[{z1,z2},{w1,w2}]**
($\mathbb{H} \rightarrow \mathbb{H}$) Returns the complex number that is the point of intersection of the geodesics whose Ideal End Points are {z1, z2} and {w1, w2}.
- **PerpendicularGeodesic[{z1, z2},{w1, w2}]**
($\mathbb{H} \rightarrow \mathbb{H}$) Returns the ideal endpoints of the geodesic perpendicular to the geodesics with endpoints {z1, z2} and {w1, w2}.

Computing with Geodesics (continued)

- **PerpendicularBisector[z1,z2]**

($\mathbb{H} \rightarrow \mathbb{H}$) Returns the ideal endpoints of the line that is the perpendicular bisector of the line segment connecting z_1 and z_2 .

Other Functions

- **Circumcenter**

Returns the circumcenter of three non-collinear points. When the input is six real numbers or three ordered pairs, Circumcenter outputs an ordered pair. When the input is three complex numbers, Circumcenter outputs a complex number.

- **Affine[List]**

Returns the inhomogeneous coordinates of the list of homogeneous coordinates of a point in projective space.

- **Projective[z]**

Returns the homogeneous coordinates of a point in P^1 corresponding to the extended complex number z .

- **Adj[M]**

Returns the adjugate of the 2×2 matrix M . $\text{Adj}[M] = \text{Det}[M] \text{Inverse}[M]$.